2013 John O'Bryan Mathematical Competition
5-person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team letter in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

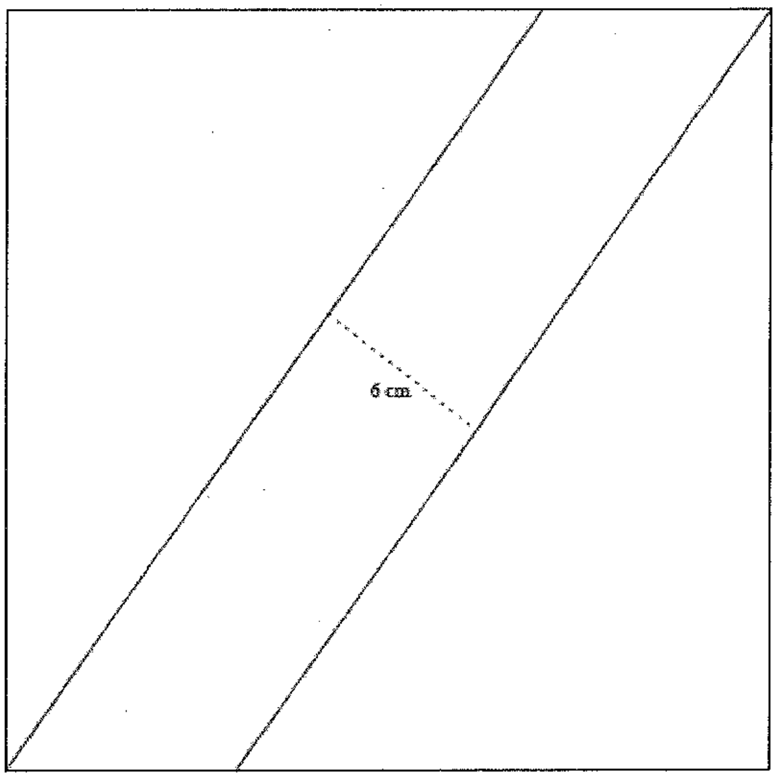
1. Let a, b, c, and d be digits with $a \neq 0$. These digits will be used to create numbers (e.g. *abcd* represents the four digit number where a is the first digit, b is the second, c is the third, and d is the fourth).

   a. Show that any two digit number *aa* is divisible by 11.

   b. Show that any six-digit number *abcabc* is divisible by 13.

   c. Show that any eight-digit number *abcdabcd* is divisible by 73.

2. A square is divided into three pieces of equal area by two parallel cuts, as shown. The distance between the parallel lines is 6 cm. What is the area of the square?



3. The Kenton County Board of Commissioners, which has 20 members, recently had to elect a President. There were three candidates: A (Alice), B (Bob), and C (Carol). On each ballot the three candidates were to be listed in order of preference, with no abstentions. It was found that 11 members (a majority?) preferred A over B (thus the other 9 preferred B over A). Likewise it was determined that 12 members preferred C over A. Given these results, it was suggested that B should withdraw to enable a runoff election between A and C. However, B protested and it was then found that 14 members preferred B over C! The Board has not yet recovered from the resulting confusion. Given that every possible order of A,B,C appeared on at least one ballot, how many board members voted for B as their first choice? How about A and C?

4. You and your opponent are about to roll dice to determine the winner of a marvelous prize. You may choose between rolling two regular six-sided dice and taking the sum; or rolling a single regular 12-sided die (with the numbers 1 to 12). Your opponent will get the dice you don't choose. If the higher roll wins the prize, should you choose the two six-sided dice or the single 12-sided die? Note: Ties are broken by rolling the dice again as necessary.

5. Recall that the formula for the degree measure of an interior (vertex) angle of a regular n-gon is $180\left(\frac{n-2}{n}\right)$

   Suppose four regular polygons, an a-gon, b-gon, c-gon, and d-gon, surround a point in a plane with no gap or overlap.

   a. Show that the following equation is satisfied: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$

   b. Find the four combinations of whole numbers that satisfy this equation.

6. In the following equation, solve for x:

$$\ln^2(x)+\ln^2(4x-5)=\ln(x^2)\ln(4x-5)$$

Solutions to these problems will be posted after the contest to:

http://math.nku.edu/job/ (or alternatively http://www.nku.edu/~math/job/)

1. Let abcd be any four-digit number. Form the 8-digit number n=abcdabcd. Show this number is always divisible by 73.

a. Show that any two-digit number aa, a≠0, is divisible by 11.

b. Show that any six-digit number abcabc, with c≠0, is divisible by 13.

c. Finally, show that any 8-digit number abcdabcd, with d≠0, is divisible by 73.

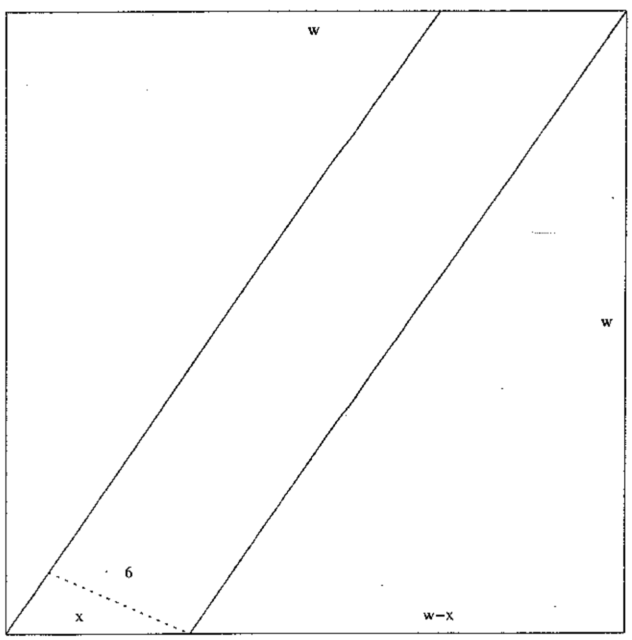
Solution :

a. aa= 10 a + a = (10+1) a = 11 a

b. abcabc=abc(1000)+abc(1)=abc(1001)=abc(13 x 17)= (17abc)(13)

c. abcdabcd=abcd(10000+1)= abcd(10001)=abcd(137)(73)

2. A square is divided into three pieces of equal area by two parallel cuts, as shown. The distance between the parallel lines is 6 cm.
What is the area of the square?



Solution: Let w be the width of the square and x be the length of the part of the square that is cut off by the two lines. Then

area of a triangle = 1/2 w (w-x) ,

but since each of the three pieces has 1/3 of the total area,

area of a triangle = 1/3 w^2.

Setting these two expressions for the area of a triangle equal and simplifying, we get w(w-3x)=0, and so w=3 x.

The side lengths of the small triangle are x, 6 and $\sqrt{x^2-36}$. Using similar triangles we get $\frac{w}{w-x} = \frac{6}{\sqrt{x^2-36}}$.

Plug 3x in for w in the similar triangles identity and simplify to get $x = \sqrt{52}$ and so $w = 3\sqrt{52}$. So the area is 9(52)=468 square centimeters.

3. The Kenton County Board of Commissioners, which has 20 members, recently had to elect a President. There were 3 candidates A (Alice), B (Bob), and C (Carol). On each ballot the 3 candidates were to be listed in order of preference, with no abstentions. It was found that 11 members (a majority ?) preferred A over B (thus the other 9 preferred B over A). Similarly it was found that 12 members preferred C over A. Given these results, it was suggested that B should withdraw to enable a runoff election between A and C. However, B protested, and it was then found that 14 members preferred B over C! The board has not yet recovered from resulting confusion. Given that every possible order of A,B,C appeared on at least one ballot, how many board members voted for B as their fist choice? How about A and C?

Solution : We use notation in the follwoing way: 7 A > B will mean that 7 voters prefered A to B.

Summarizing: 11 A > B
12 C>A
14 B > C

Since there were only 20 voters, there were 3 voters who voted both A > B and C > A. We can do similarly for each pair of preferences giving

3 C > A > B
5 A > B > C
6 B > C > A

In addition each preference appeared on at least one ballot:

3 C > A > B
5 A > B > C
6 B > C > A
1 A > C > B
1 B > A > C
1 C > B > A

This is a total of 17 ballots, with 9 A > B, 10 C > A and 12 B > C. With these 3 remaining votes we need to add two more voters to each of these preferences. We list how each preference ballot changes the three pairs of interest:

3 C > A > B	C>A, A>B
5 A > B > C	A>B,B>C
6 B > C > A	B>C, C>A
1 A > C > B	A>B
1 B > A > C	B>C
1 C > B > A	C>A

Since we need to change a total of 6 pairs, adding to any of the last 3 preferences won't work. But note that we cannot add two more to any one of the first 3 preferences as that will increase 2 of the pairs by 2, and trying to increase the remaining preference will also increase another, making it too large. So the only option is to add one to each of the first three preferences, Then 7+1=8 voters listed B first. (7 A and 5 C)

Comment: While a few teams came up with a correct solution, they did not explain how they found their solution (or why if was the only one.) Clear, well-explained work is required for full credit on these problems.

4. You and your opponent are about to roll dice to determine the winner of a marvelous prize. You may choose between rolling two regular six-sided dice and taking the sum; or rolling a single twelve-sided die (with numbers 1-12.) Your opponent will get the dice you don't choose. If the higher roll wins the prize, should you choose the two six-sided dice, or the single 12-sided die?
(Note: In case of a tie, you will split the prize.)

Solution :

Suppose the 12-sided die rolls 3. Then for the 12-sided
to win the two six-sided rolled a 2. One chance out of 36 of this happening.
to tie the two six-sided rolled a 3 as well. There are two chances out of 36 of this happening.
to lose there are 33 out of 36 chances.
Repeating this for all possible rolls gives a nice table

Roll of 12 sided	Prob on 2 - sided	12 - sided wins	Tie	2 Six - sided win
1	0	0	0	36
2	1	0	1	35
3	2	1	2	33
4	3	3	3	30
5	4	6	4	26
6	5	10	5	21
7	6	15	6	15
8	5	21	5	10
9	4	26	4	6
10	3	30	3	3
11	2	33	2	1
12	1	35	1	0

Notice that for rows 2-12 there is a beautiful symmetry with each player winning half the time. Since the two six-sided win the first row, there is a slightly better chance of winning with the 2 six-sided dice than with the one 12-sided die.

(Problem can also be done using probability computations.)

Note : Several teams pointed out that the expected roll value for the two 6-sided dice (7) is higher that that of the one 12-sided die (6.5). While this is true, and would make you suspect that the two one-sided dice are better, it is not enough. Is is possible for a doe with a higher expected value to lose over 50% of the time to one with a lower expected value. These teams were given partial credit on the problem.

5. Recall that the formula for the measure of an interior (vertex) angle of a regular n - gon is $\frac{n-2}{n}$ 180.Suppose four regular polygons, an a - gon, b - gon, c - gon, and d - gon, surround a point in a plane with no gap or overlap

a.Show that the following equation is satisfied. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$

b..Find the four combinations of whole numbers that satisfy this equation.

Solution:

a. Since the sum of the angles around any point add to 360 degrees we get that

$\frac{a-2}{a}$ 180+$\frac{b-2}{b}$ 180+$\frac{c-2}{c}$ 180+$\frac{d-2}{d}$ 180 =360. dividing by 180 and rewritting we have

$\left(1-\frac{2}{a}\right)+\left(1-\frac{2}{b}\right)+\left(1-\frac{2}{c}\right)+\left(1-\frac{2}{d}\right)=2.$ Subtracting two from each side and adding each $\frac{2}{\text{variable}}$ term we then have

$2=\frac{2}{a}+\frac{2}{b}+\frac{2}{c}+\frac{2}{d}$ which is equivalent to $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1.$

b. The solutions a=b=c=d=1/4 and a=b=3, c=d =6 are easy to find. These both have combinations that add to 1/2 in different ways. Combining the two ideas gives a=3, b=6, c=d=1/4.

The final solution can be found looking at a common denominator of 12 (fractions of form k/12). The numerators could be any of 1,2,3 4, or 6 (to give a fraction of the form 1/x) Looking for a combination that adds to 12 we get 4,4,3,1, giving fractions of 1/3,1/3,1/4, and 1/12. So the last solution is a=b=3,c=4, d=12.

6. Solve for x: $\ln^2(x) + \ln^2(4x-5) = \ln(x^2)\ln(4x-5)$

Solution :

Rewrite as $\ln^2(x) + \ln^2(4x-5) = 2\ln(x)\ln(4x-5)$ or, rewritten,
$\ln^2(x) - 2\ln(x)\ln(4x-5) + \ln^2(4x-5) = 0$, which is a perfect square,

$(\ln x - \ln(4x-5))^2 = 0.$

This implies that ln x = ln(4x-5) and since the log is a one-to-one function, x=4x-5 or x=5/3.

Caution: Be careful not to make up your own log properties. These are important - learn them cold!