2018 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**.

- Place your team code in the upper right corner of each page that will be turned in.
- Place problem numbers in the upper left corner (failure to do these things will result in no score for that problem/page).

Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

1. Given a square of unit area:

   a. Show that the square can be partitioned into six squares (Note: The squares do NOT need to be congruent.)
   b. For $n = 2, 3, 4$, and 5, is it possible to partition the square into n squares? Explain your reasoning (and for any case where it is possible, show how).
   c. Show that the square can always be partitioned into n squares for $n > 6$.

2. **Partition** the set $U = \{1,2,3,4,5,6,7,8,9,10\}$ into two non-empty subsets, S and P, in such a way that the **sum** of the numbers in S is equal to the **product** of the numbers in P. Note that the product or sum of a single number is considered to be that number itself.

   a. Find a solution.
   b. Find a second solution.
   c. Find all solutions and explain how you know that your list of solutions is complete.

3. Given the following set of symbols: $\{1,2,3,4,+,-,*\}$ where * represents multiplication. You may create expressions using each of these symbols **exactly once** together with any number of parentheses.

   a. What is the maximum value your expression can attain?
   b. What is the minimum value your expression can attain?
   c. What is the minimum that the absolute value of your expression can attain?

4. A circle has both an inscribed and circumscribed regular polygon (both having the same number of sides). Find the ratio of areas for the larger polygon to the smaller:

   a. If the polygon is a triangle.
   b. If the polygon is a square.
   c. If the polygon is a hexagon.
   d. If the polygon is has n sides. As n gets large, what number does the ratio approach?

5. Jayden and Cody decide to play a coin flipping game. They decide to flip a fair coin until they obtain a sequence of either five consecutive heads or five consecutive tails, at which point the game will end.

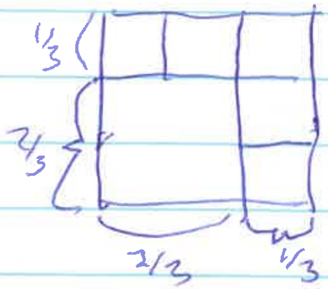
   a. What is the probability the game ends within the first five flips?
   b. What is the probability the game ends within the first six flips?
   c. What is the probability the game ends within the first seven flips?
   d. Suppose they instead use a biased coin (i.e. a coin for which the two sides have unequal probability). Would using the game be expected to end sooner or would it likely be prolonged by the use of such a coin. Justify your answer.

6. Consider the equation $y^2 = x^2 + b$ where x and y are positive integers.

   a. Find all solutions (x, y) if $b = 24$.
   b. Find all solutions (x, y) if $b = 60$. Explain how you know that you have all solutions.
   c. Show that there are no solutions (x, y) if $b = 210$.

Problem #1

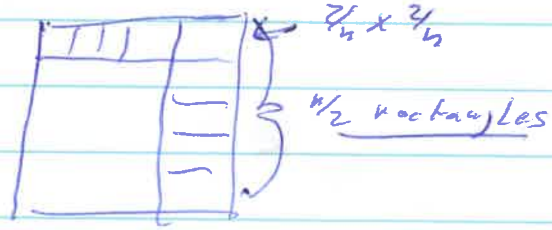
1 a)



b) $n = 2, 3,$ and 5 are not possible

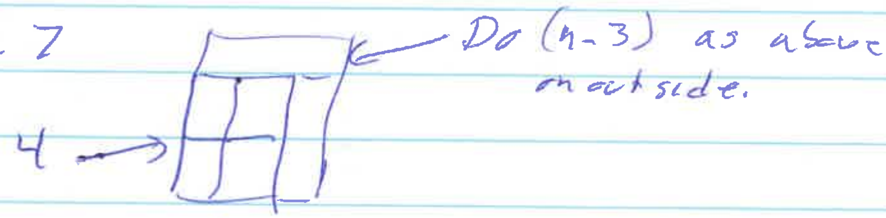
$n = 4$ Obvious:

c) Any even number ≥ 6



$$\underbrace{\left(\frac{n}{2} + \frac{n}{2} - 1\right)}_{\text{corner counted twice!}} + 1 = n$$

For odd ≥ 7



Problem #2

(2) Let p be the product of elements in P and s be the sum of elts in $\bar{P}$. Then since the sum of all ten numbers is $\frac{10(11)}{2} = 55$, we have that $p = 55 - s$ or $p + s = 55$ for numbers to have same sum and product.

Note that $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 > 55$ & so no product that satisfies the equation, above, will have more than 4 numbers. Also there cannot be only one number in the product ($x + x = 55$ has no solution for $x \leq 10$.)

(The solutions are $P = \{6, 7\}$, $P = \{1, 4, 10\}$ and $P = \{1, 2, 3, 7\}$)

2 elts: (Try out cases. Call second element y.)

$x = 1$ $\quad 1 \cdot y + (1 + y) = 55 \Rightarrow 2y = 54$ No Solution

$x = 2$ $\quad 2 \cdot y + (2 + y) = 55 \Rightarrow 3y = 53$ " "

etc → to $x = 5$ no sol'n.

$x = 6$ $\quad 6 \cdot y + (6 + y) = 55 \Rightarrow 7y = 49$ $\quad y = 7$

$x = 7$ $\quad 7y + (7 + y) = 55$ $\quad 8y = 48$ $\quad y = 6$ (repeat!)

$x = 8$ $\quad 8y + (8 + y) = 55$ $\quad 9y = 47$ No sol'n

$x = 9$ $\quad 9y + (9 + y) = 55$ $\quad 10y = 46$ "

$x = 10$ $\quad 10y + (10 + y) = 55$ $\quad 11y = 45$ "

(+) (2 elts: (6, 7) works.)

Check $6 \cdot 7 = 42 = 1 + 2 + 3 + 4 + 5 + 8 + 9 + 10$

3 elts: $P = \{x, y, z\}$ We need $xyz + (x + y + z) = 55$

If x, y, z all odd, then xyz is also odd. But 4 odds can't sum to 55.

If one elt, say x, is even, then xyz is even and so $xyz + (x+y+z) = 55$ is only possible if two of x and y are even, one odd.

Try 2 evens:

$2 \cdot 4 \cdot z + (2+4+z) = 55$ $\quad 9z = 49$ No Sol'n

$2 \cdot 6 z + (2+6+z) = 55$ $\quad 13z = 47$ " "

$2 \cdot 8 z + (2+8+z) = 55$ $\quad 17z = 45$ " "

$2 \cdot 10 z + (2+10+z) = 55$ $\quad 21z = 43$ " "

$4 \cdot 6 z + (4+6+z) = 55$ $\quad 25z = 45$ " "

$4 \cdot 8 z + (4+8+z) = 55$ $\quad 33z = 43$ " "

* $4 \cdot 10 z + (4+10+z) = 55$ $\quad 41z = 41$ $\quad z=1$

$6 \cdot 8 z + (6+8+z) = 55$ $\quad 49z = 41$ No Sol'n

$6 \cdot 10 z + \longrightarrow$ $\quad 61z = 55 -$ " "

For 3 elements $(1, 4, 10)$

4 elts $\quad P = \{x, y, z, w\}$ $\quad x \cdot y \cdot z \cdot w + (x+y+z+w) = 55$

$1 \cdot 2 \cdot 3 \cdot 4 + (1+2+3+4) = 55$ $\quad 24 + 10 \neq 55$ No

$1 \cdot 2 \cdot 3 \cdot 5 + (1+2+3+5) = 55$ $\quad 30 + 11 \neq 55$ No

$1 \cdot 2 \cdot 3 \cdot 6 + (1+2+3+6) = 55$ No

* $\boxed{1 \cdot 2 \cdot 3 \cdot 7} + (1+2+3+7) = 42 + 13 = 55$ Yes

$1 \cdot 2 \cdot 3 \cdot x$ for $x \geq 8$ will be too large! No

$1 \cdot 2 \cdot 4 \cdot 5 + (1+2+4+5) = 40 + 12$ No

$1 \cdot 2 \cdot 4 \cdot 6 + (1+2+4+6) = 48 + 13 = 61$ No

$1 \cdot 2 \cdot 4 \cdot x$ Too large No.

$1 \cdot 3 \cdot 4 \cdot 5 + (1+3+4+5) = 60 +$ No

Anything else is too large.

4 elts only $(1, 2, 3, 7)$ works.

Problem #3

3) $\{1, 2, 3, 4, +, -, *\}$ and parentheses

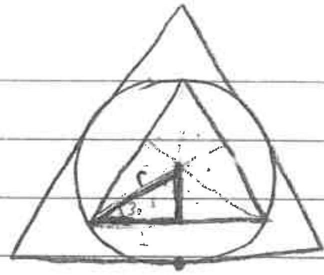
a) Max: $19 = 4*(3+2)-1$

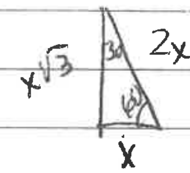
b) Min: $-19 = 1 - 4*(3+2)$

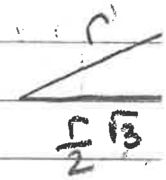
c) min abs value: $(1+2-3)*4 = 0$.

Problem #4

a.



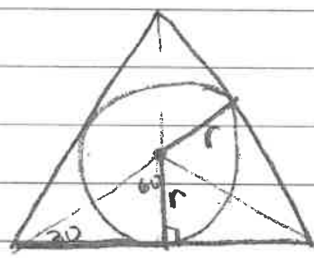


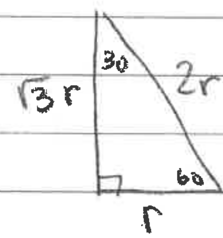


Small Δ $A = 6\left(\frac{1}{2}bh\right) = 6\left(\frac{1}{2}\left(\frac{\sqrt{3}r}{2}\right)\left(\frac{r}{2}\right)\right)$

$= \frac{3\sqrt{3}r^2}{4}$

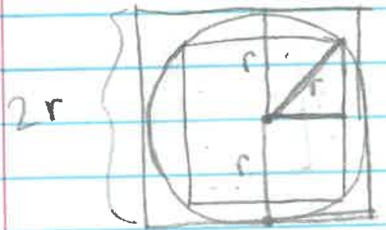
large Δ $A = 6\left(\frac{1}{2}bh\right) = 6\left(\frac{1}{2}(\sqrt{3}r)(r)\right) = 3\sqrt{3}r$



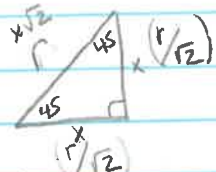


ratio: $\frac{\text{large}}{\text{small}} = \frac{3\sqrt{3}r^2}{\frac{3\sqrt{3}r^2}{4}} = \boxed{4}$

Square



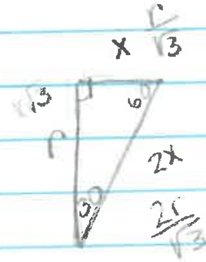
Large $\square$ Area $= (2r)^2 = 4r^2$

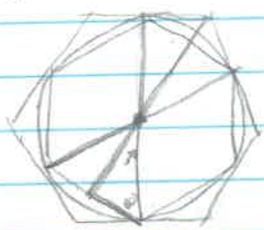


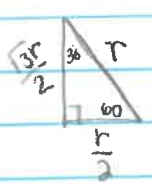
Small $\square$ Area $= 8\left(\frac{1}{2}\left(r/\sqrt{2}\right)\left(r/\sqrt{2}\right)\right) = \dfrac{4r^2}{2} = 2r^2$

$$\frac{\text{Large}}{\text{Small}} = \frac{4}{2} = \boxed{2}$$

hexagon







large hexagon area =

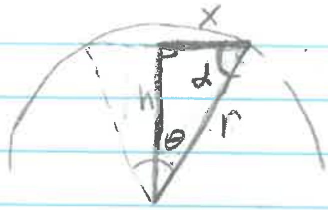
$A = 12\left(\frac{1}{2}\left(r \cdot \frac{r}{\sqrt{3}}\right)\right) = \dfrac{6r^2}{\sqrt{3}}$

Small hexagon area $\quad A = 12\left(\frac{1}{2}\left(\frac{r}{2}\right)\frac{\sqrt{3}r}{2}\right)$

$A = \dfrac{3\sqrt{3}\,r^2}{2}$

$$\frac{\text{Large}}{\text{Small}} = \frac{\overset{2}{6}r^2}{\sqrt{3}} \cdot \frac{2}{3\sqrt{3}r^2} = \boxed{\frac{4}{3}} \checkmark$$

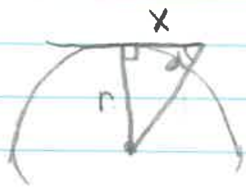
general



Small n-gon $A = \frac{1}{2}(x \cdot h) = \frac{1}{2}(r\sin\alpha)(r\cos\alpha) = \frac{1}{2}r^2\sin\alpha\cos\alpha$

$\sin\alpha = \frac{h}{r}$ $\qquad$ $\cos\alpha = \frac{x}{r}$

$h = r \cdot \sin\alpha$ $\qquad$ $x = r \cdot \cos\alpha$



Large n-gon $A = \frac{1}{2}(x \cdot r) = \frac{1}{2}\left(\frac{r}{\tan\alpha}\right)(r)$

$\tan\alpha = \frac{r}{x}$ $\qquad$ $= \frac{r^2}{2\tan\alpha}$

$x = \frac{r}{\tan\alpha}$

$$\frac{\text{Large}}{\text{Small}} = \frac{\frac{r^2}{2\tan\alpha}}{\frac{1}{2}r^2\sin\alpha\cos\alpha} = \frac{r^2\cos\alpha}{2\sin\alpha} \cdot \frac{2}{r^2\sin\alpha\cos\alpha}$$

$$= \frac{1}{\sin\alpha}$$

As $n \to \infty$, $\alpha \to \frac{\pi}{2}$

$$= \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = \boxed{1} \checkmark$$

Problem # 5

5) 5 Heads or 5 tails in a row.

a) 5H or 5T out of 5 rolls

$$\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^4 = 1/16$$

or

$$1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/16$$

any same same same same

b) Prob (out of 6) = Pr(1st 5) + Pr(last 5)

$$\frac{1}{16} + 2 \cdot \left(\frac{1}{2}\right)^6 = \frac{1}{16} + \frac{1}{32} = 3/32$$

H TTTTT or T HHHHH

from b,

c) $3/32 + 4 \cdot \left(\frac{1}{2}\right)^7$

HT 5H
HH 5T
TH 5T
TT 5T

d) If we repeat part a) with probabilities of 1/3 and 2/3, we get

$$\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)^5 = \frac{1+32}{243} = \frac{33}{243} > 1/16$$

So more likely to end earlier.

Problem #6

6) $y^2 = x^2 + b$ can be written as

$y^2 - x^2 = (y-x)(y+x) = b$

a) $b = 24$ $(y-x)(y+x) = 24$

Factor pairs for 24: $(1\cdot 24)$ $(2\cdot 12)$ $(3\cdot 8)$ $(4\cdot 6)$

$(2,12)$ $y = 7,\ x = 5$ The others have no

$(4,6)$ $y = 5,\ x = 1$ integer solutions.

b) $b = 60$ $(y-x)(y+x) = 60$

$1\cdot 60 / 2\cdot 30 / 3\cdot 20 / 4\cdot 15 / 5\cdot 12 / 6\cdot 10$

$2\cdot 30$ $y = 16,\ x = 14$ No solutions for

$6\cdot 10$ $y = 8,\ x = 2$ the other factor pairs.

c) $b = 210$ Here $210 = 2\cdot 3\cdot 5\cdot 7$

no matter how you factor, one in the pair will be even & one will be odd & so, as above, there are no solutions.