2024 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper. Questions will not be scored without the following two items:

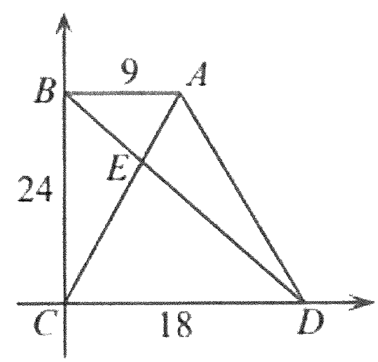
- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. **Teams must show complete solutions (not just answers) to receive credit.** More specific instructions are read verbally at the beginning of the test.

1. The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle C defined by $x^2+y^2=100$.

   a. Determine the equation of the perpendicular bisector of segment AB.
   b. Determine the point(s) of intersection of circle C and the perpendicular bisector of segment AB. Label these points P and Q.
   c. Determine the length of segment PQ and simplify your answer.

2. In the diagram below, ∠ABC and ∠BCD are right angles. Also, AB = 9, BC = 24, and CD =18. The diagonals (segments AC and BD) of quadrilateral ABCD intersect at point E



   a. Determine the ratio DE:EB.
   b. Determine the area of triangle DEC.
   c. Determine the area of triangle DAE.

3. For this problem set notation is not required, but you must clearly define the required sets.

   a. Determine all real solutions to the equation $a^2+10=a+(10)^2$
   b. Determine the set of all ordered pairs (m,n) such that

      $m \neq n$ and $m^2+n=m+n^2$

   c. In terms of only *a*, determine the set of all real number triples (*a*, *b*, *c*) such that

      $a^2+b^2+c=b^2+c^2+a=c^2+a^2+b$

4. Brooke, a statistician, places numbers on a 3-by-3 grid using the following rule, called Brooke's Principle:

    For any three adjacent numbers in a horizontal, vertical, or diagonal line, the middle number is always the arithmetic average (mean) of its two neighbors.

   a. Using Brooke's Principle, determine the missing numbers in the Grid A below.
   b. Determine, with appropriate supporting work, the **sum** of the nine numbers in Grid B below, when it is completed using Brooke's Principle.
   c. Determine, with appropriate supporting work, the values of x and y in Grid C below, when it is completed using Brooke's Principle.

GRID A

3		19
8		

GRID B

k		
5		23

GRID C

x	7	
9		y
		20

5. Define the following two infinite sequences for all non-negative integers m and n, with $0 \le n \le m$:

$$\text{Sequence(1): } A_{m-n} + A_{m+n} = \frac{1}{2}A_{(2m)} + \frac{1}{2}A_{(2n)}$$

$$\text{Sequence(2): } B_{m-n} + B_{m+n} = B_{(2m)} + B_{(2n)}$$

   a. Using Sequence(1), find the value of A_0
   b. Determine the values of A_2 and A_3 in Sequence(1), if $A_1 = 1$.
   c. Determine the value of B_{2024} in Sequence(2), if $B_0 = 5$

6. Let f be a function on the set of real numbers such that the following <u>condition</u> is true for all real numbers x:

$$f\left(x^2\right)+\left(f(x)\right)^2=6$$

   a. Find all constant functions $f(x)$ for which this condition holds.

   b. Consider all functions of the form $f(x)=cx$ where c is an integer. Find all values of c so that the solution to $f\left(x^2\right)+\left(f(x)\right)^2=6$ is an integer.

   c. For all functions $f(x)$ satisfying the condition, find all possible values of $f(0)+f(1)$.

2024 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required and it is advisable to leave them off. Each problem has the same point-value.

1. Find the **ordered pair** that represents the point at which the line with equation of $3y - 4x = 21$ intersects the y-axis. Give your answer as an ordered pair.

2. A bag of marbles contains 6 red, 8 blue, and 7 yellow marbles. A marble is drawn at random from the bag. Find the probability that a yellow marble is drawn. Express your answer as a common fraction reduced to lowest terms.

3. A right triangle has vertices $(-3, 5)$, $(2, 4)$, and $(0, -6)$. Find the exact length of the hypotenuse. Give your answer as a simplified radical expression (e.g. $a\sqrt{b}$).

4. For what value of k is the following true: $(1)^k = (3)^k = (6)^k$

5. A square inscribed in a circle has area 80. Find the area of the circle. Give an exact answer in terms of π.

6. The measure of the central angle that intercepts an arc of length 11π in a circle with area 289π is k degrees. Find k. Express your answer as a decimal rounded to the nearest hundredth.

7. A boss wants to give all of his employees a bonus of \$100 for the holidays, but would be \$11 short. The boss compromises by giving each of his employees \$98 as a bonus. This resulted in the boss having \$23 left over. How many employees does the boss have?

8. There is a circle inscribed inside a right triangle whose legs have length 10 and 30. Find the length of the circle's radius. Give your answer with two decimal places.

9. Find the length of the chord connecting the points of intersection for two circles whose equations are $(x-8)^2 + (y+7)^2 = 100$ and $(x+4)^2 + (y+7)^2 = 100$.

10. If $\frac{2}{3}x$ is 10% of 180, find the value of 10% of $\frac{4}{9}x$. Express your answer as a decimal.

11. Find the value of k if $(\sqrt[4]{3})(\sqrt[3]{2}) = \sqrt[12]{k}$.

12. A regular polyhedron has 6 distinct vertices and 12 distinct edges. If the length of one of these edges is 8, find the total surface area of this regular polyhedron. Give your answer as a simplified radical expression (e.g. $a\sqrt{b}$).

13. Triangle ΔABC has area 4224 square units. The midpoints of the three sides are joined to form ΔDEF. The midpoints of the three sides of ΔDEF are then joined to form ΔXYZ. Find the area of ΔXYZ.

14. One hour after leaving port, a boat develops engine trouble which slows its speed to 75% of its usual rate. Continuing at this constant rate, the boat arrives at its destination 1 hour and 45 minutes later than it would have by traveling at its usual rate. If the engine troubles would have occurred 50 miles later, the boat would have reached its destination 1 hour and 30 minutes later than it would have by traveling at its usual rate. Find the number of miles between the port and its destination, rounded to the nearest integer.

15. A convex polygon has consecutive vertices $(2,13), (1,14), (3,16), (6,17), (7,16)$ and $(5,13)$. Find the area of this polygon. Give your answer as a reduced improper fraction

16. When $x^3 + x + k$ is divided by $(x + 5)$, the remainder is 24. Find the value of k.

17. Consider the equation below. Find the value of k.

$$2\left(\frac{1}{4}\right)^{(2k-3)} = \frac{1}{32}$$

18. A right cylindrical tube has an outside circumference of 18, and has a height of 17. Eight turns of a wire are helically wound from top to bottom in such a way that the ends of the wire are vertically aligned on the side of the tube. Find the length of the wire.

19. $\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} = \frac{1 + \sqrt{157}}{2}$. Find the value of x.

20. Given triangle ΔETS with $E(0,0)$, $T(10,0)$ and $S(6,6)$. Let R be the point where the angle bisector of $\angle EST$ intersects $\overline{ET}$. Let K be the centroid of ΔETS. Find the length of $\overline{RK}$. Express your answer as a decimal rounded to 4 significant digits.

Name: ______________________ Team Code: _______

2024 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________ 11. ______________________

2. ______________________ 12. ______________________

3. ______________________ 13. ______________________

4. ______________________ 14. ______________________

5. ______________________ 15. ______________________

6. ______________________ 16. ______________________

7. ______________________ 17. ______________________

8. ______________________ 18. ______________________

9. ______________________ 19. ______________________

10. ______________________ 20. ______________________

2024 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required and it is advisable to leave them off. Each problem has the same point-value.

1. Find the **y-coordinate** of which the vertical line passing through (3,4) intersects the parabola whose equation is $y = x^2 - x - 6$.

2. Let b represent the degree measure of an angle such that $\tan(b) = \frac{\sqrt{3}}{3}$. If $180 < b < 270$, find the value of b.

3. A bag contains only green, yellow, blue, and orange marbles. The bag contains 17 green marbles, 24 yellow marbles, fewer than 13 blue marbles, and x orange marbles. If the probability that a marble drawn at random from the bag is orange is exactly 25%, find the sum of all possible distinct values of x.

4. Let $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ represent vectors such that $\vec{a} = (5,2)$, $\vec{b} = (-1,-3)$, and $\vec{c} = (-8,5)$ $\vec{a} = (5,2), \vec{b} = (-1,-3)$, and $\vec{c} = (-8,5)$. Find the **ordered pair** representing $\vec{d}$ if $\vec{a} + \vec{b} = \vec{c} - \vec{d}$.

5. Let a, b, c, and d be single digit positive integers such that $f(x) = ax^3 + bx^2 + cx + d$. If $f(5) = 518$ *and* $f(10) = 3483$, find the value of $(5a + 2b + 4c + d)$.

6. Find the value of the indicated sum: $\sum_{k=2}^{4}\left(5k + 3^k\right)$

7. Let b and x represent real numbers and let k be an element of the set $\{1, 3, 5, 7, 9\}$. Find the sum of all distinct values of k such that $(b^x)(b^k) = \left(b^{(x+k)}\right)$.

8. Let k represent the degree measure of an angle such that $\sin(59°) = \cos(k°)$. If $270° < k < 360°$, find the value of k.

9. If $2024! - 2023! = (2022!)x^2$, find the value of x.

10. If $x \neq 0$ and $\frac{x^k}{x^{(5(k+1))}} = x^3$, find the value of k.

11. A right cylindrical tube has an outside circumference of 18, and has a height of 17. Eight turns of a wire are helically wound from top to bottom in such a way that the ends of the wire are vertically aligned on the side of the tube. Find the length of the wire.

12. When $(-3x + 5y)^6$ is expanded and completely simplified, the coefficient of one of the terms is 84,375. Find the exponent of y for that term.

13. Let $i = \sqrt{-1}$. Find the complex conjugate of $17 - 8i^7$.

14. Find the value of $\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n\right]$. Give your answer as a fraction reduced to lowest terms.

15. Let $i = \sqrt{-1}$ and let a, b, and c represent positive integers. Find the smallest possible value of $(a + b - c)$ for which $\left|\frac{3i}{2+6i}\right| = \frac{a\sqrt{b}}{c}$.

16. In taking an eleven-problem multiple choice test with four choices for each problem, a student randomly guesses on all eleven questions. Find the probability that the student guessed at most six correct answers out of the eleven. Express your answer as a decimal rounded to the nearest ten-thousandth.

17. Let m be a positive integer such that $1 + 3 + 6 + \cdots + \frac{m(m+1)}{2} = 9(1 + 2 + 3 + \cdots + m)$. Find the value of m.

18. The sum of the last two terms of an eight-term geometric progression of real terms is $\frac{2}{9}$. The sum of the third and fourth term of this geometric progression is 18. Find the sum of all eight terms of this geometric progression. Express your answer as an improper fraction reduced to lowest terms.

19. Given triangle ΔETS with $E(0,0)$, $T(10,0)$ and $S(6,6)$. Let R be the point where the angle bisector of $\angle EST$ intersects $\overline{ET}$. Let K be the centroid of ΔETS. Find the length of $\overline{RK}$. Express your answer as a decimal rounded to 4 significant digits.

20. A convex polygon has consecutive vertices $(2,13), (1,14), (3,16), (6,17), (7,16)$ and $(5,13)$. Find the area of this polygon. Write your answer as a reduced improper fraction.

Name: ______________________ Team Code: _______

2024 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________
2. ______________________
3. ______________________
4. ______________________
5. ______________________
6. ______________________
7. ______________________
8. ______________________
9. ______________________
10. ______________________
11. ______________________
12. ______________________
13. ______________________
14. ______________________
15. ______________________
16. ______________________
17. ______________________
18. ______________________
19. ______________________
20. ______________________

2024 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

*****Calculators may not be used on the first four questions*****

1. Let $f(x) = 4x^2 - 5x + 3$ and $g(x) = 3x - 6$. Find the value of $f\big(g(4)\big) + g(f(4))$.

2. An ellipse has an equation of $\frac{(x-4)^2}{256} + \frac{(y+3)^2}{361} = 4$. The area of this ellipse can be expressed in the form of $n\pi$. Find the value of n.

3. Let *r* be the numerical remainder when dividing $t^4 + 4t^3 - 6t^2 + 31t - 15$ by $(t - 4)$. For all real numbers *x* and *y*, $f(x + y) = f(x) + f(y)$. If $f(1) = 5$, let $f(7) = k$. Find the value of $(r + k)$.

4. $\begin{bmatrix} x & -2 \\ 4 & y \end{bmatrix}\begin{bmatrix} 4 & 1 \\ y & x \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 20 & 10 \end{bmatrix}$. Find the value of $x - y$.

*****Calculators may be used on the remaining questions*****

5. A sum of \$5,000 is invested at an interest rate of 10% annual percentage rate. If compounded quarterly, in how many years will that sum be worth at least \$24,000.

6. Let k be the sum of all positive integers *x* such that $(x - 2)(x^2 - 15x + 32) < 0$. Let w be the length of a radius of a circle whose circumference contains the same number of units as its area contains square units. Find the value of $(k + w)$.

7. Let k and w be positive integers where $k > w$. Find the number of distinct ordered pairs (k, w) that exist such that $\sqrt{k} + \sqrt{w} = 27$

8. Let the volume of a cone with its height equal to its radius be 576π. The area of its base is $k\pi$. Let the diagonal of a cube be of length 15. The area of one of its sides is *j*. Find the product *jk*.

Tiebreaker 1. If x is an integer such that $7 \le x \le 21$, find the sum of all distinct values for x such that 441_x is the square of an integer.

Tiebreaker 2. Find the value of $\sum_{k=1}^{\infty}\left(45\left(\frac{1}{3}\right)^k\right)$. Give your answer as an exact decimal.

Tiebreaker 3. Find the sum of the solutions to the equation $x^3 + 4x^2 - 9x - 36 = 0$.

Names: ______________________________ **School:**______________________

2024 John O'Bryan Mathematical Competition
Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

Calculators are not allowed to be used on the first four questions!

		SCORE
1.	__________________________	_____
2.	__________________________	_____
3.	__________________________	_____
4.	__________________________	_____
5.	__________________________	_____
6.	__________________________	_____
7.	__________________________	_____
8.	__________________________	_____
T1.	__________________________	
T2.	__________________________	

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

Name: ______ANSWERS______ Team Code: _______

2024 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1.	(0,7)	Must be this ordered pair	11.	432	
2.	1 / 3	Must be this fraction	12.	$200\sqrt{3}$	Must be in this form
3.	$\sqrt{130}$ or $1\sqrt{130}$	Must be in this form	13.	264	
4.	0		14.	417	
5.	40π	Must be in this form	15.	29 / 2	Must be this improper fraction
6.	116.47	Must be this exact decimal	16.	154	
7.	17		17.	3	
8.	4.19	Must be this exact decimal	18.	145	
9.	16		19.	39	
10.	1.2	Must be this exact decimal	20.	2.001	Must be this decimal

Awards Lists and Solutions to the Team Competition may be found at http://math.nku.edu/job

Name: ______ANSWERS______ Team Code: _______

2024 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 0
2. 210
3. 62
4. (–12,6) — Must be this ordered pair.
5. 58
6. 162
7. 25
8. 329
9. 2023
10. –2
11. 145
12. 4
13. $17 - 8i$ or $-8i + 17$
14. 1/2 — Must be this fraction.
15. –7
16. 0.0076 — Must be this decimal.
17. 25
18. 1640 / 9 — Must be this improper fraction.
19. 2.001 — Must be this decimal.
20. 29/2 — Must be this improper fraction.

Names: ______________________________ **School:**______________________

2024 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

Calculators are not allowed to be used on the first four questions!

	Answer	SCORE
1.	252	____
2.	1216	____
3.	560	____
4.	1	____
5.	16	____
6.	78	____
7.	13	____
8.	10800	____
T1.	210	____
T2.	22.5 **Must be this decimal**	____

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

T3. -4

SCORE

2024 John O'Bryan Mathematics Competition
5-Person Team Test
SOLUTIONS

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper. Questions will not be scored without the following two items:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. **Teams must show complete solutions (not just answers) to receive credit.** More specific instructions are read verbally at the beginning of the test.

1. The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle C defined by $x^2+y^2=100$.

   a. Determine the equation of the perpendicular bisector of segment AB.

   The slope of AB is
   $(6-(-8))/(-8-(-6)) = -7$
   So the slope of the perpendicular bisector is 1/7.

   The perpendicular bisector of AB contains the midpoint of segment AB, which is $(-7, -1)$.

   So the equation of the perpendicular bisector is $y + 1 = (1/7)(x + 7)$, or $y = (1/7)x$. Note that $y = x/7$ is also acceptable.

   b. Determine the point(s) of intersection of circle C and the perpendicular bisector of segment AB. Label these points P and Q.

   From part (a), $x = 7y$. Substituting this expression into the equation of the circle yields

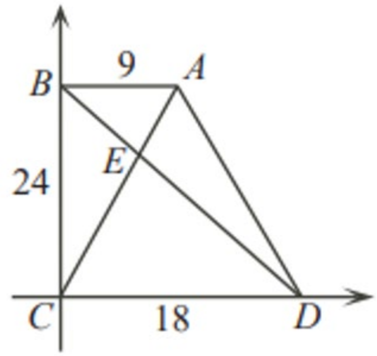
   $$(7y)^2+y^2=100$$
   $$50y^2=100$$
   $$y^2=2$$
   $$y=\pm\sqrt{2}$$

   Thus the points of intersection are $\left(7\sqrt{2},\sqrt{2}\right)$ and $\left(-7\sqrt{2},-\sqrt{2}\right)$.

   c. Determine the length of segment PQ and simplify your answer.

   Since $y = (1/7)x$ contains (0,0), which is the center of circle C, segment PQ is a diameter of circle C. Since the radius of the circle is 10, the length of PQ is 20.

2. In the diagram below, ∠ABC and ∠BCD are right angles. Also, AB = 9, BC = 24, and CD =18. The diagonals (segments AC and BD) of quadrilateral ABCD intersect at point E



a. Determine the ratio DE:EB.

Since segments BA and CD are parallel, angles ABD and BDC are congruent. Angles BEA and DEC are congruent vertical angles. So, triangles ABE and CDE are similar, by the Angle-Angle theorem. Thus:

DE/BE = CD/AB = 18/9 = 2/1

So, DE:EB = 2:1

b. Determine the area of triangle DEC.

From part (a), triangles ABE and CDE are similar, with sides in the ratio 1:2. This also implies that the altitude of triangle CDE is twice that of triangle ABE. The sum of the altitudes of triangles CDE and ABE is 24 units. So the altitude of triangle CDE is (2/3)(24) = 16 units. This makes the area of triangle DEC is (1/2)(18)(16) = 144 square units.

c. Determine the area of triangle DAE.

The altitude of triangle ACD to base CD is 24. So, the area of triangle ACD is (1/2)(18)(24) = 216 square units. The area of triangle DEC from (b) is 144 square units. The area of triangle AED is the difference in these areas: 216 – 144 = 72 square units.

3. For this problem set notation is not required, but you must clearly define the required sets.

a. Determine all real solutions to the equation $a^2+10=a+(10)^2$

Algebraic steps as follows:

$$a^2+10=a+(10)^2$$
$$a^2-(10)^2=a-10$$
$$(a+10)(a-10)-(a-10)=0$$
$$(a-10)(a+10-1)=0$$
$$(a-10)(a+9)=0$$

From this either a = 10 or a = –9.

b. Determine the set of all ordered pairs (m,n) such that

$m \neq n$ and $m^2+n=m+n^2$

Similar logic is used:

$$m^2+n=m+n^2$$
$$m^2-n^2=m-n$$
$$(m+n)(m-n)-(m-n)=0$$
$$(m-n)(m+n-1)=0$$

Since m and n may not be equal, it must be the case that $m + n - 1 = 0$. Thus any values if m and n satisfying $m + n = 1$ will satisfy the system.

c. In terms of only a, determine the set of all real number triples (a, b, c) such that

$$a^2+b^2+c=b^2+c^2+a=c^2+a^2+b$$

Applying a similar logic as in part (b):

$$a^2+b^2+c=b^2+c^2+a$$
$$a^2-c^2=a-c$$
$$(a+c)(a-c)-(a-c)=0$$
$$(a-c)(a+c-1)=0$$

This means either $a = c$ or $a = 1 -$ c. Similarly equating first and third expressions yields $b = c$ or $b = 1 - c$, while equating the second and third yields $a = b$ or $b = 1 - a$. We want everything in terms of a. Possible ordered triples must therefore be of one of these forms:

(a,a,a), $(a,a,1-a)$, $(a,1-a,a)$, or $(a,1-a,1-a)$

4. Brooke, a statistician, places numbers on a 3-by-3 grid using the following rule, called Brooke's Principle:

For any three adjacent numbers in a horizontal, vertical, or diagonal line, the middle number is always the arithmetic average (mean) of its two neighbors.

GRID A

3		19
8		

GRID B

k		
5		23

GRID C

x	7	
9		y
		20

a. Using Brooke's Principle, determine the missing numbers in the Grid A below.

The average of 3 and 19 is (1/2)(3 + 19) = 11, which goes between 3 and 19 on the top row. The number 8 is the average of 3 and some other number, or 8 = (1/2)(3 + x). Solving this equation yields x = 13, Using similar reasoning on the remaining cells (which the students should provide), the completed grid is given below:

3	11	19
8	16	24
13	21	29

b. Determine, with appropriate supporting work, the **sum** of the nine numbers in Grid B below, when it is completed using Brooke's Principle.

The average of 5 and 23 is (1/2)(5 + 23) = 14, Since the average of the numbers in the "5" column equals 5, the sum of the numbers in the column is 3(5) = 15. Likewise the sum of the numbers in the "14" column is 3(14) = 42 and the sum of the numbers in the "23" column is 3(23) = 69. So, the sum of the 9 numbers in the table is 15 + 42 + 69 = 126.

c. Determine, with appropriate supporting work, the values of x and y in Grid C below, when it is completed using Brooke's Principle.

The center square is the average of 9 and y AND the average of x and 20. This yields the equation $(1/2)(x + 20) = (1/2)(9 + y)$, or $x - y = -11$.

The square in the top right corner gives an average of 7 when added to x, so its value must be $2(7) - x = 14 - x$). This square likewise gives an average of y when added to 20, so its value must also equal $2y - 20$. This yields the equation $14 - x = 2y - 20$, or $x + 2y = 34$.

Solving the system of equations (students should do this) yields $x = 4$ and $y = 15$.

5. Define the following two infinite sequences for all non-negative integers m and n, with $0 \le n \le m$:

$$\text{Sequence(1): } A_{m-n} + A_{m+n} = \frac{1}{2}A_{(2m)} + \frac{1}{2}A_{(2n)}$$

$$\text{Sequence(2): } B_{m-n} + B_{m+n} = B_{(2m)} + B_{(2n)}$$

a. Using Sequence(1), find the value of A_0

To find A_0, let m = n = 0. With these values, $A_0 + A_0 = (1/2)A_0 + (1/2)A_0$, which simplifies to $2A_0 = A_0$. Hence $A_0 = 0$.

b. Determine the values of A_2 and A_3 in Sequence(1), if $A_1 = 1$.

Since $A_0 = 0$ and $A_1 = 1$, taking $m = 1$ and $n = 0$, we get

$$A_1 + A_1 = \frac{1}{2}A_2 + \frac{1}{2}A_0$$

$$1 + 1 = \frac{1}{2}A_2 + 0$$

This yields $A_2 = 4$

To find, A_3, let $m = 2$ and $n = 1$. With these values,

$$A_1 + A_3 = \frac{1}{2}A_4 + \frac{1}{2}A_2$$

So, A_4 is needed to find A_3. To find A_4, let m = 2 and n = 0. Then

$$A_2 + A_2 = \frac{1}{2}A_4 + \frac{1}{2}A_0$$

$$4 + 4 = \frac{1}{2}A_4 + 0$$

So $A_4 = 16$. Returning to the earlier equation,

$$A_1 + A_3 = \frac{1}{2}A_4 + \frac{1}{2}A_2$$

$$1 + A_3 = \frac{1}{2}(16) + \frac{1}{2}(4)$$

So $A_3 = 9$

c. Determine the value of B_{2024} in sequence (2), if $B_0 = 5$

To find B_{2024}, let $m = n = 1012$. With these values, $B_0 + B_{2024} = B_{2024} + B_{2024}$

So $B_{2024} = B_0 = 5$.

6. Let f be a function on the set of real numbers such that the following condition is true for all real numbers x:

$$f\left(x^2\right)+\left(f\left(x\right)\right)^2=6$$

a. Find all constant functions $f\left(x\right)$ for which this condition holds.

Let $f\left(x\right)=k$, where k is some constant. Then the above condition becomes:

$$k+k^2=6$$
$$k^2+k-6=0$$
$$\left(k+3\right)\left(k-2\right)=0$$

Thus $f\left(x\right)=-3$ and $f\left(x\right)=2$ both satisfy the condition.

b. Consider all functions of the form $f\left(x\right)=cx$ where c is an integer. Find all values of c so that the solution to $f\left(x^2\right)+\left(f\left(x\right)\right)^2=6$ is an integer.

Since $f\left(x\right)=cx$, $f\left(x^2\right)=cx^2$. Thus $cx^2+c^2x^2=6$ which reduces to

$$x^2=\frac{6}{c^2+c}$$

For x to be an integer, $c^2+c=6$. So $c=-3$ or $c=2$. (as in part (a))

Note: One might note that x^2 would be an integer if c^2+c totaled 3, 2, or 1. This is true, but x would not be a perfect square under any of those conditions. So the only integer denominator that works is 6. Fractional denominators (e.g. 1/6) could work, but for none of those would the value if c be an integer.

c. For all functions $f\left(x\right)$ satisfying the condition, find all possible values of $f\left(0\right)+f\left(1\right)$.

We start with $f\left(x^2\right)+f\left(x\right)=6$ and note that this means $f\left(0\right)+\left[f\left(0\right)\right]^2=6$.
So applying part (a), $f\left(0\right)=-3$ or $f\left(0\right)=2$.

Likewise, $f\left(1\right)+\left[f\left(1\right)\right]^2=6$, so $f\left(1\right)=-3$ or $f\left(1\right)=2$.

So, f(0) + f(1) could equal any of the following:

$-3+(-3)=-6$ $\quad$ $-3+2=-1$ $\quad$ $2+2=4$

To show these values are attainable, consider the following functions:

- If $f\left(x\right)=-3$, then $f(0)+f(1)=-6$.
- If $f\left(x\right)=2$, then $f(0)+f(1)=4$.
- If $f\left(x\right)=-3$ for all $x\neq 0$ and $f\left(0\right)=2$, then $f(0)+f(1)=-1$ for all values of x.